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Extensions of Some Fixed Point Theorems of Rhoades

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In [4, 10, 12, 14, and 15], it has been shown that for a mapping T satisfying a certain contractive condition, if the sequence of Mann iterates converges it converges to a fixed point of T.

In this paper it is shown that, for mappings T which satisfy conditions (I) or (II) below, if the sequence of Ishikawa iterates converges, it converges to the fixed point of T. These results extend the corresponding results of Rhoades [12], and Hicks and Kubicek [4].

Let X be a Banach space and C be a nonempty subset of X. Let $T: C \to C$ be a mapping. The iteration scheme called I-*scheme* is defined as follows:

$$x_0 \in C, \tag{1}$$

$$y_n = \beta_n Tx_n + (1-\beta_n) x_n, \qquad n \geqslant 0, \tag{2}$$

$$x_{n+1} = (1-\alpha_n) x_n + \alpha_n Ty_n, \quad n \geqslant 0. \tag{3}$$

In the Ishikawa scheme, $\{\alpha_n\}$, $\{\beta_n\}$ satisfy $0 \leqslant \alpha_n \leqslant \beta_n \leqslant 1$ for all n, $\lim_{n\to\infty} \beta_n = 0$, and $\sum \alpha_n \beta_n = \infty$. In this paper we shall make the assumption that (i) $0 \leqslant \alpha_n, \beta_n \leqslant 1$ for all n, (ii) $\underline{\lim}\, \alpha_n = \alpha > 0$, (iii) $\overline{\lim}\, \beta_n = \beta < 1$.

The two contractive conditions to be used are the following. There exists a constant k, $0 \leqslant k < 1$ such that for all $x, y \in X$,

(I) $\|Tx - Ty\| \leqslant k \max\{\|x-y\|, \|x-Tx\|, \|y-Ty\|, \|x-Ty\| + \|y-Tx\|\}$.

0022-247X/83 $3.00

(II) At least one of the following conditions holds:

(A) For each $x, y \in X$, $\|x - Tx\| + \|y - Ty\| \leqslant a\|x - y\|$, $1 \leqslant a < 2$;

(B) For each $x, y \in X$, $\|x - Tx\| + \|y - Ty\| \leqslant b[\|x - Ty\| + \|y - Tx\| + \|x - y\|]$, $\frac{1}{2} \leqslant b < \frac{2}{3}$;

(C) For each $x, y \in X$, $\|x - Tx\| + \|y - Ty\| + \|Tx - Ty\| \leqslant c[\|x - Ty\| + \|y - Tx\|]$, $1 \leqslant c < \frac{3}{2}$;

(D) For each $x, y \in X$, $\|Tx - Ty\| \leqslant k \max\{\|x - y\|, \|x - Tx\|, \|y - Ty\|, [\|x - Ty\| + \|y - Tx\|]/2\}$, $0 \leqslant k < 1$.

DEFINITION 1.1. Let X be a Banach space. A mapping $T: X \to X$ is called a *quasicontraction* if there exists a constant k, $0 \leqslant k < 1$ such that for each $x, y \in X$,

$$\|Tx - Ty\| \leqslant k \max\{\|x - y\|, \|x - Tx\|, \|y - Ty\|, \|x - Ty\|, \|y - Tx\|\}.$$

THEOREM 1.2. *Let X be a normed linear space and C be a closed convex subset of X. Let $T: C \to C$ be a mapping satisfying* (I), *$\{x_n\}$ the sequence of the* I*-scheme associated with T and such that $\{\alpha_n\}$ is bounded away from zero. If $\{x_n\}$ converges to p, then p is a fixed point of T.*

Proof. It follows from (3) that $x_{n+1} - x_n = \alpha_n(Ty_n - x_n)$. Since $x_n \to p$, $\|x_{n+1} - x_n\| \to 0$. Since $\{\alpha_n\}$ is bounded away from zero, $\|Ty_n - x_n\| \to 0$. It also follows that $\|p - Ty_n\| \to 0$. Since T satisfies (I) we have

$$\|Ty_n - Tx_n\| \leqslant k \max\{\|y_n - x_n\|, \|x_n - Tx_n\|, \|y_n - Ty_n\|, \|x_n - Ty_n\| + \|y_n - Tx_n\|\};$$

$$\begin{aligned}\|y_n - x_n\| &= \|\beta_n Tx_n + (1 - \beta_n) x_n - x_n\| \leqslant \beta_n \|x_n - Tx_n\| \\ &\leqslant \|x_n - Tx_n\| \leqslant \|x_n - Ty_n\| + \|Ty_n - Tx_n\|;\end{aligned}$$

$$\begin{aligned}\|y_n - Ty_n\| &= \|\beta_n Tx_n + (1 - \beta_n) x_n - Ty_n\| \leqslant \beta_n \|Tx_n - Ty_n\| \\ &\quad + (1 - \beta_n)\|x_n - Ty_n\| \\ &\leqslant \|x_n - Ty_n\| + \|Tx_n - Ty_n\|;\end{aligned}$$

$$\begin{aligned}\|y_n - Tx_n\| &= \|\beta_n Tx_n + (1 - \beta_n) x_n - Tx_n\| \\ &\leqslant \|x_n - Ty_n\| + \|Ty_n - Tx_n\|.\end{aligned}$$

Thus

$$\|Tx_n - Ty_n\| \leqslant \frac{2k}{1-k} \|x_n - Ty_n\|.$$

Taking the limit as $n \to \infty$ we obtain

$$\|Tx_n - Ty_n\| \to 0.$$

It follows that $\|x_n - Tx_n\| \leqslant \|x_n - Ty_n\| + \|Ty_n - Tx_n\|$ and $\|p - Tx_n\| \leqslant \|p - x_n\| + \|x_n - Tx_n\|$ tend to zero as $n \to \infty$.

Using the definition of T and the triangle inequality we have

$$\begin{aligned} \|Tx_n - Tp\| \leqslant k \max\{&\|x_n - p\|, \|x_n - Tx_n\|, \|x_n - p\| \\ &+ \|x_n - Tx_n\| + \|Tp - Tx_n\|, \|p - Tx_n\| \\ &+ \|x_n - Tx_n\| + \|Tx_n - Tp\|\}. \end{aligned}$$

Taking the limit as $n \to \infty$ we obtain $\|Tx_n - Tp\| \to 0$. Finally

$$\|p - Tp\| \leqslant \|p - Tx_n\| + \|Tx_n - Tp\| \to 0,$$

showing that $p = Tp$.

The above theorem is a generalization of Theorem 9 [13] as can be seen from the following example.

EXAMPLE 1.3. Let $X = R$ with the usual metric. Define $T: X \to X$ by $Tx = x$. Then clearly T satisfies (I). Indeed, $\|x - Ty\| + \|y - Tx\| = 2\|x - y\|$, $\|Tx - Ty\| = \|x - y\|$. Let $k = \frac{2}{3}$. Then $\|Tx - Ty\| \leqslant k[\|x - Ty\| + \|y - Tx\|]$. However, T is not a quasicontraction.

THEOREM 1.4. *Let X be a Banach space, T a self-mapping of X satisfying (II). Let $\{x_n\}$ be the sequence of the I-scheme satisfying* (i), (ii), *and* (iii). *If* $\{x_n\}$ converges, then it converges to a fixed point of T.

Proof. As in the proof of Theorem 1.2 it follows that $\|Ty_n - x_n\|$ and $\|p - Ty_n\|$ tend to zero as $n \to \infty$.

Proof. If II(A) is satisfied, then

$$2\|Tx_n - Ty_n\| \leqslant a\|x_n - y_n\| + \|x_n - Ty_n\| + \|y_n - Tx_n\|:$$

$$\|y_n - x_n\| \leqslant \beta_n[\|x_n - Ty_n\| + \|Ty_n - Tx_n\|], \tag{4}$$

$$\|y_n - Tx_n\| \leqslant (1 - \beta_n)[\|x_n - Ty_n\| + \|Ty_n - Tx_n\|]. \tag{5}$$

Thus

$$[1+\beta_n(1-a)]\|Tx_n-Ty_n\| \leqslant [2+\beta_n(a-1)]\|x_n-Ty_n\|.$$

If II(B) is satisfied, then

$$2\|Tx_n-Ty_n\| \leqslant (1+b)[\|x_n-Ty_n\|+\|y_n-Tx_n\|] + b\|y_n-x_n\|.$$

Using (4) and (5) we obtain

$$[1-b+\beta_n]\|Tx_n-Ty_n\| \leqslant [2(1+b)-\beta_n]\|x_n-Ty_n\|.$$

If II(C) is satisfied, then

$$3\|Tx_n-Ty_n\| \leqslant (c+1)[\|x_n-Ty_n\|+\|y_n-Tx_n\|].$$

Using (5) we obtain

$$[2-c+(1+c)\beta_n]\|Tx_n-Ty_n\| \leqslant (1+c)(2-\beta_n)\|x_n-Ty_n\|.$$

If II(D) is satisfied, then

$$\|Tx_n-Ty_n\| \leqslant k\max\{\|x_n-y_n\|, \|x_n-Tx_n\|, \|y_n-Ty_n\|, [\|x_n-Ty_n\|+\|y_n-Tx_n\|]/2\}.$$

$$\begin{aligned}\tfrac{1}{2}[\|x_n-Ty_n\|+\|y_n-Tx_n\| &\leqslant \tfrac{1}{2}[\|x_n-Ty_n\|+\|x_n-Ty_n\|+\|Ty_n-Tx_n\| \\ &= \|x_n-Ty_n\|+\tfrac{1}{2}\|Tx_n-Ty_n\|.\end{aligned}$$

Again using (4) and (5) we have

$$\|Tx_n-Ty_n\| \leqslant \frac{k}{1-k}\|x_n-Ty_n\|.$$

Thus, in any case

$$\|Tx_n-Ty_n\| \leqslant \max\left\{\frac{2+\beta_n(a-1)}{1+\beta_n(1-a)}, \frac{2(1+b)-\beta_n}{1-b+\beta_n}, \frac{1+c(2-\beta_n)}{2-c+(1+c)\beta_n}, \frac{k}{1-k}\right\}\|x_n-Ty_n\|. \tag{6}$$

Recalling that $\overline{\lim}\ \beta_n=\beta<1$, and taking the limit in (6) yields

$$\|Tx_n-Ty_n\| \to 0 \quad \text{as} \quad n\to\infty.$$

Therefore, $\|x_n - Tx_n\|$ and $\|p - Tx_n\|$ tend to zero as $n \to \infty$.

$$\|p - Tp\| \leqslant \|p - x_n\| + \|x_n - Tx_n\| + \|Tx_n - Tp\|. \tag{7}$$

If x_n, p satisfy II(A), then

$$\begin{aligned}\|Tx_n - Tp\| &\leqslant \tfrac{1}{2}[a\|x_n - p\| + \|x_n - Tp\| + \|p - Tx_n\|] \\ &\leqslant \tfrac{1}{2}[a\|x_n - p\| + \|x_n - Tx_n\| + \|Tx_n - Tp\| \\ &\quad + \|p - Tx_n\|].\end{aligned}$$

If x_n, p satisfy II(B), then

$$\begin{aligned}2\|Tx_n - Tp\| &\leqslant (1+b)[\|x_n - Tp\| + \|p - Tx_n\|] + b\|x_n - p\| \\ &\leqslant (1+b)[\|x_n - Tx_n\| + \|Tx_n - Tp\| \\ &\quad + \|p - Tx_n\|] + b\|x_n - p\|.\end{aligned}$$

If x_n, p satisfy II(C), then

$$\|Tx_n - Tp\| \leqslant \frac{(c+1)}{2-c}[\|x_n - Tx_n\| + \|p - Tx_n\|].$$

If x_n, p satisfy II(D), then

$$\begin{aligned}\|Tx_n - Tp\| &\leqslant k \max\{\|x_n - p\|, \|x_n - Tx_n\|, \|p - Tp\|, \\ &\quad \tfrac{1}{2}[\|x_n - Tp\| + \|p - Tx_n\|\} \\ &\leqslant k \max\{\|x_n - p\|, \|x_n - Tx_n\|, \|p - x_n\| \\ &\quad + \|x_n - Tx_n\| + \|Tx_n - Tp\|, \tfrac{1}{2}[\|x_n - Tx_n\| \\ &\quad + \|Tx_n - Tp\| + \|p - Tx_n\|]\}.\end{aligned}$$

Hence,

$$\|Tx_n - Tp\| \leqslant \frac{k}{1-k}[\|x_n - p\| + \|p - Tx_n\| + \|x_n - Tx_n\|].$$

Substituting the value of $\|Tx_n - Tp\|$ in (7) we have

$$\begin{aligned}\|p - Tp\| \leqslant \|p - x_n\| + \|x_n - Tx_n\| + \max\left\{\frac{a}{2-a}, \frac{1+b}{1-b},\right. \\ \left.\frac{c+1}{2-c}, \frac{k}{1-k}\right\}[\|x_n - p\| \\ + \|p - Tx_n\| + \|x_n - Tp\|] + \|x_n - Tx_n\|.\end{aligned} \tag{8}$$

Taking the limit in (8) as $n \to \infty$, we have $p = Tp$.

DEFINITION 2.1. Let X be a normed linear space and C be a nonempty subset of X. A mapping $T: C \to C$ is called *strictly pseudocontractive* if for some k, $0 \leqslant k < 1$, and all $x, y \in C$,

$$\|Tx - Ty\|^2 \leqslant \|x - y\|^2 + k\,\|(I - T)x - (I - T)y\|^2.$$

T is called *pseudocontractive* if for all $x, y \in C$,

$$\|Tx - Ty\|^2 \leqslant \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2.$$

T is called *demicontractive* if for some k, $0 \leqslant k < 1$, for all $x \in C$ and $y \in F(T) = \{x \in C : Tx = x\}$,

$$\|Tx - Ty\|^2 \leqslant \|x - y\|^2 + k\,\|x - Tx\|^2.$$

T is called *hemicontractive* if for all $x \in C$ and $y \in F(T)$,

$$\|Tx - Ty\|^2 \leqslant \|x - y\|^2 + \|x - Tx\|^2.$$

T is said to satisfy *condition* (T) (after Tricomi) if for all $x \in C$ and $y \in F(T)$

$$\|Tx - y\| \leqslant \|x - y\|.$$

Clearly any strictly pseudocontractive mapping is hemicontractive, any mapping satisfying condition (T) is demicontractive and a demicontractive mapping is hemicontractive, but not conversely.

EXAMPLE 2.2. Let $X = R$ with the absolute value norm and $C = [0, 1]$. Define $T: C \to C$ by $Tx = \frac{1}{4}$, $0 \leqslant x \leqslant \frac{1}{3}$, $Tx = 0$, $\frac{1}{3} < x \leqslant 1$. Then T is demicontractive with fixed point $\frac{1}{4}$. However, T does not satisfy condition (T). Indeed, let $x = \frac{2}{5}$, $y = \frac{1}{4}$. Then $|Tx - y| = |0 - \frac{1}{4}| = \frac{1}{4} > \frac{3}{20} = |x - y|$.

EXAMPLE 2.3 ([13, p. 748]). Let $X = R^2$ with the usual norm. Let $C = [-1, 1] \times [-1, 1]$. Define $T: C \to C$ by $T(x, y) = (-y, x)$. Then T is pseudocontractive, but not strictly pseudocontractive.

EXAMPLE 2.4. Let $X = R$ with the absolute value norm and $C = [0, 1]$. Define $T: C \to C$ by $Tx = \frac{1}{2}$, $0 \leqslant x \leqslant \frac{1}{2}$, $Tx = 0$, $\frac{1}{2} < x \leqslant 1$. Then clearly T is hemicontractive with fixed point $y = \frac{1}{2}$. Since $k < 1$, it follows that $(k + 1)^{-1} > \frac{1}{2}$. Choose $\frac{1}{2} < x < (k + 1)^{-1}$, $y = \frac{1}{2}$. Then $|Tx - Ty|^2 = \frac{1}{4}$, whereas $|x - y|^2 + k\,|x - Tx|^2 = (x - \frac{1}{2})^2 + kx^2 < \frac{1}{4}$.

THEOREM 2.5. *Let H be a Hilbert space and C be a convex subset of H. Suppose $T: C \to C$ satisfies condition (T). Suppose $F(T)$ is nonempty. Suppose $\sum \alpha_n \beta_n$ diverges and $\beta_n \to \beta < 1$. Then $\underline{\lim}\, \|x_n - Tx_n\| = 0$ for each $x_0 \in C$, where x_{n+1} is defined as in the I-scheme.*

Proof. Since T satisfies (T), T is demicontractive for any constant k. Choose $k < 1-\beta$. Then T is demicontractive with contractive constant k. For any x, y, z in a Hilbert space and real number λ, we have

$$\|\lambda x + (1-\lambda)y - z\|^2 = \lambda\|x-z\|^2 + (1-\lambda)\|y-z\|^2 - \lambda(1-\lambda)\|x-y\|^2.$$

Thus for $p \in F(T)$ and each integer n,

$$\begin{aligned} 0 \leqslant \|x_{n+1}-p\|^2 &= \|\alpha_n Ty_n + (1-\alpha_n)x_n - p\|^2 \\ &= (1-\alpha_n)\|x_n-p\|^2 + \alpha_n\|Ty_n-p\|^2 \\ &\quad - \alpha_n(1-\alpha_n)\|x_n-Ty_n\|^2 \\ &\leqslant (1-\alpha_n)\|x_n-p\|^2 + \alpha_n\|Ty_n-p\|^2. \end{aligned}$$

Using property (T) we have

$$\|x_{n+1}-p\|^2 \leqslant (1-\alpha_n)\|x_n-p\|^2 + \alpha_n\|y_n-p\|^2.$$

Using demicontractiveness of T and definition of y_n we have

$$\|y_n-p\|^2 \leqslant \|x_n-p\|^2 - \beta_n(1-\beta_n-k)\|x_n-Tx_n\|^2.$$

Hence,

$$\begin{aligned} 0 \leqslant \|x_{n+1}-p\|^2 &\leqslant (1-\alpha_n)\|x_n-p\|^2 + \alpha_n\|x_n-p\|^2 \\ &\quad - \alpha_n\beta_n(1-\beta_n-k)\|x_n-Tx_n\|^2 \\ &= \|x_n-p\|^2 - \alpha_n\beta_n(1-\beta_n-k)\|x_n-Tx_n\|^2. \end{aligned} \tag{9}$$

By induction, we obtain

$$0 \leqslant \|x_0-p\|^2 - \sum_{i=0}^{n} \alpha_i\beta_i(1-\beta_i-k)\|x_i-Tx_i\|^2.$$

Thus

$$\sum_{n=0}^{\infty} \alpha_n\beta_n(1-\beta_n-k)\|x_n-Tx_n\|^2 \leqslant \|x_0-p\|^2. \tag{10}$$

First we note that $\beta_n \geqslant \beta_n(1-\beta_n)$ $(0 \leqslant \beta_n \leqslant 1)$. Let $\eta = 1-\beta-k$. Then $\eta > 0$, and there exists an integer N such that $\beta_n < \beta + \eta/2$ for all $n \geqslant N$. Thus $1-\beta_n-k > 1-k-\beta-\eta/2 = \eta/2$. Therefore, $\sum \alpha_n\beta_n(1-\beta_n-k) \geqslant \eta/2 \sum \alpha_n\beta_n$, which diverges. Hence $\sum \alpha_n\beta_n(1-\beta_n-k)$ diverges. Thus from (7) we obtain $\underline{\lim}\,\|x_n-Tx_n\| = 0$.

Remark 2.6. In Theorem 2.5, if $\beta \neq 0$, $\{\alpha_n\}$ remains bounded away from 0, then the terms of the series $\sum \alpha_n\beta_n(1-\beta_n-k)$ are bounded away from zero. Hence we conclude that $\lim \|x_n-Tx_n\| = 0$.

Remark 2.7. Since $1-\beta_n-k \to 1-\beta-k>0$, there exists an integer N_0 such that $1-\beta_n-k>0$ for all $n \geqslant N_0$. Thus from Eq. (9) we obtain $\|x_{n+1}-p\| \leqslant \|x_n-p\|$ for $n \geqslant N_0$.

Before we state and prove our final result, we need to recall the following lemma.

OPIAL'S LEMMA [8]. *Suppose H is a Hilbert space and the sequence $\{x_n\}$ is weakly convergent to x_0. Then for any $x \neq x_0$, $\underline{\lim}\, \|x_n-x_0\| < \underline{\lim}\, \|x_n-x\|$.*

THEOREM 2.8. *Let H be a Hilbert space and C be a closed convex subset of H. Suppose $T: C \to C$ such that*

(a) $F(T) \neq \varnothing$.

(b) *T satisfies condition (T).*

(c) *If any sequence $\{x_n\}$ converges weakly to x and $(I-T)\,x_n$ converges strongly to 0, then $(I-T)\,x=0$.*

Then for any $x_1 \in C$ and $\beta_n \to \beta$, $0<\beta<1-k$ the I-scheme converges weakly to a fixed point of T.

Proof. Let $p \in F(T)$. By Remark 2.7 there exists an integer N such that $\|x_{n+1}-p\| \leqslant \|x_n-p\|$ for all $n \geqslant N$. If $x_N=p$ then clearly $x_n \to p$. If $x_N \neq p$, $\|x_N-p\|=r>0$. Let $S_r(p)=\{x \in H: \|x-p\| \leqslant r\}$ and let $D=C \cap S_r(p)$, then $\{x_n\}_{n=N}^{\infty} \subset D$. Also D is weakly compact, being closed and convex. Thus there exists a subsequence $\{x_{n_j}\}$ which converges weakly to $y \in D \subset C$. By Remark 2.6 $(I-T)\,x_{n_j} \to 0$, hence by condition (c), $(I-T)\,y=0$, i.e., $Ty=y$.

Suppose $\{x_n\}$ does not converge weakly to y. Then the sequence $\{x_n\}_{n=N}^{\infty}$ has at least one other weak cluster point $z \neq y$. Suppose $\{x_{m_i}\}$ converges weakly to z. As for y, $Tz=z$. It follows from Remark 2.7 that the sequences $\{\|x_n-y\|\}$ and $\{\|x_n-z\|\}$ are nonincreasing for sufficiently large n. Thus $\underline{\lim}\, \|x_n-y\|$ and $\underline{\lim}\, \|x_n-z\|$ both exist. Using the Opial's Lemma we get the following contradiction

$$\begin{aligned}
\underline{\lim}_{n}\, \|x_n-y\| &= \underline{\lim}_{j}\, \|x_{n_j}-y\| < \underline{\lim}_{j}\, \|x_{m_j}-z\| \\
&= \underline{\lim}_{i}\, \|x_{m_i}-z\| < \underline{\lim}_{i}\, \|x_{m_i}-y\| \\
&= \underline{\lim}_{n}\, \|x_n-y\|.
\end{aligned}$$

Therefore $\{x_n\}$ converges weakly to $y \in F(T)$.

The following problems which arose in the study of this paper remain unanswered.

PROBLEM. Does the Mann iteration process converge for continuous hemicontractive or Lipschitzian hemicontractive mappings?

PROBLEM. Can the Mann and Ishikawa iteration procedures be extended to a class of functions larger than Lipschitzian pseudocontractive mappings?

PROBLEM. Can the Ishikawa iteration procedure be extended to demicontractive, and quasicontraction mappings?

Let X be a Banach space. A mapping $T: X \to X$ is said to satisfy *condition* (B) if there exist constants c, k, $0 \leqslant k < 1$, $c > 0$ such that for all $x, y \in X$ we have

$$\|Tx - Ty\| \leqslant k \max\{c\,\|x-y\|, [\|x-Tx\| + \|y-Ty\|] \times [\|x-Ty\| + \|y-Tx\|]\}.$$

The following result, which is a slight generalization of [3, Theorem 3], can be easily proved.

Let C be a closed convex subset of a Banach space X, T be a self-mapping of C satisfying condition (B), x_n be the sequence of Mann iterates associated with T, where $\{\alpha_n\}$ satisfies (i) $\alpha_0 = 1$, (ii) $0 < \alpha_n \leqslant 1$ for $n > 0$, and (iii) $\underline{\lim}_n \alpha_n = \alpha > 0$. If $\{x_n\}$ converges in C, then it converges to a fixed point.

PROBLEM. Does the same conclusion hold if the Mann iteration procedure is replaced by that of Ishikawa iteration procedure?

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